Ex: Does $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Sol: If we plug in the x- or y-axis, the top becomes zero, so the limits along the axes are zero. However, along $y=x$:

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

So, the limit does not exist. ◇

A convenient trick to use to show limits DO NOT exist is to check along the line $y=mx$, where m is arbitrary. (This is for limits to $(0,0)$.) If there is a dependence on m, the limit does not exist. Consider again the last example. Along $y=mx$ we have

$$\lim_{(x,mx)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

Since m is arbitrary, this limit does not exist.

This trick, however, is not cure-all.

Ex: Does $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

Sol: along $y=mx$: $\lim\limits_{(x,mx)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim\limits_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim\limits_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0.$

But, along $x = y^2$,

$$\lim_{(y^2,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2} \neq 0.$$

So, the limit does not exist. ◊

So, when can we take limits?

<u>Def</u>: A function $f(x,y)$ is <u>continuous at (a,b)</u> if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We say f is <u>continuous on D</u> if it is continuous at every point in D.

<u>Facts for Checking for Continuity</u>:

- Polynomials are continuous
- Sums of continuous functions are continuous
- Products " " " " "
- Quotients " " " " "
  when the denominator is $\neq 0$.
- The composition of continuous functions is continuous.

<u>Ex</u>: Does $\lim_{(x,y)\to(0,0)} \frac{4-xy}{x^2+3y^2}$ exist?

<u>Sol</u>: $f(x,y) = \frac{4-xy}{x^2+3y^2+1}$ is continuous at $(0,0)$ since it is a quotient of polynomials and the denominator isn't zero at $(0,0)$.

Thus the limit exists and its value is $f(0,0) = \frac{4-0}{0+0+1} = 4$ ◇

Ex: Determine where $g(x,y) = \ln(x^2+y^2-4) - \sqrt{y}$ is continuous.

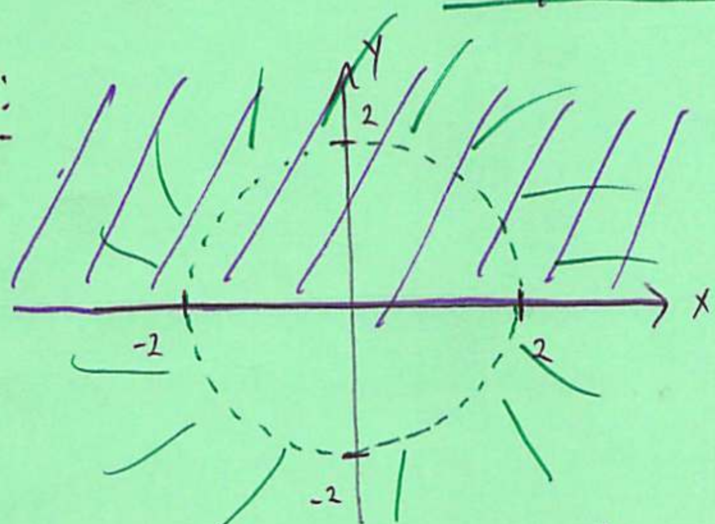
Sol: 1) $\ln(t)$ is continuous for $t>0$

2) $\ln(x^2+y^2-4)$ is continuous for $x^2+y^2-4>0$

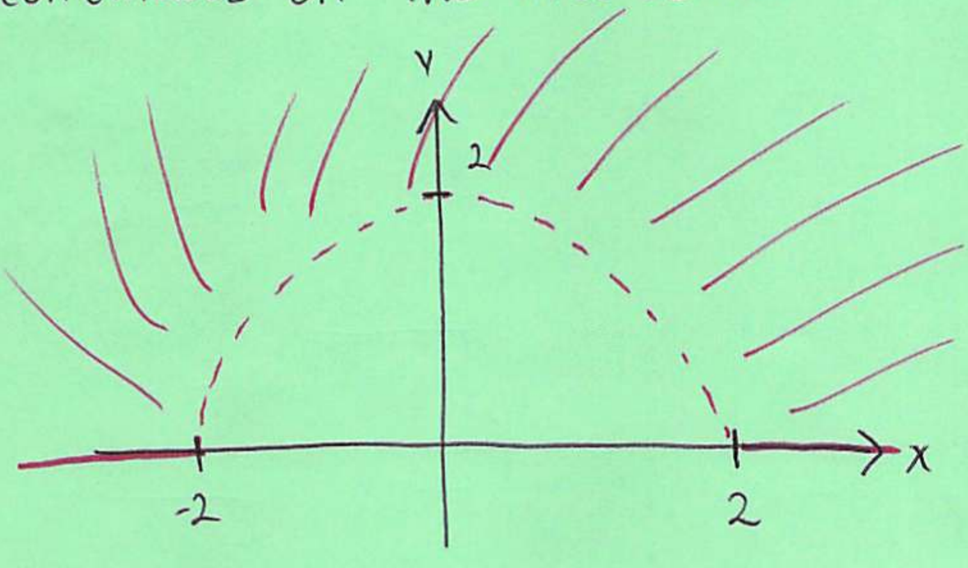
3) $\sqrt{y}$ is continuous for $y \geq 0$.

So, g is continuous when $\underline{x^2+y^2-4>0}$ and $\underline{y \geq 0}$

Graphically:



g is continuous on the intersection of these two regions:



◇

14.3 - Partial Derivatives

Def: The partial derivative of $f = f(x,y)$

- with respect to x is

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

- with respect to y is

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Naturally, these are not the most practical way to compute partial derivatives. Notice that, for example in $\frac{\partial f}{\partial x}$, the limit has no concern for y. This means we can compute $\frac{\partial f}{\partial x}$ by pretending y is just a constant and taking the derivative with respect to x. Likewise for finding $\frac{\partial f}{\partial y}$.

Ex: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x,y) = x^3 + 2x^2y + x^4y^2 + \sqrt{y}$

Sol:

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy + 4x^3y^2, \quad \frac{\partial f}{\partial y} = 2x^2 + 2x^4y + \frac{1}{2} \cdot \frac{1}{\sqrt{y}}$$

◇

What are the values of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to be interpreted as? They are the <u>rates of change of f</u> in the x- and y-directions, respectively. Visually, we see this number as the slope of the curve on the surface, passing through the point in either the x- or y-direction. (See Mathematica code for visuals.)

<u>Ex</u>: Find the first partials of $f(x,y) = \sin(x\cos y)$.

<u>Sol</u>:
$$\frac{\partial f}{\partial x} = \cos(x\cos y)\cdot\left(\frac{\partial}{\partial x}(x\cos y)\right)$$
$$= \cos(x\cos y)\cos y$$
$$\frac{\partial f}{\partial y} = \cos(x\cos y)\left(\frac{\partial}{\partial y}(x\cos y)\right) = -\cos(x\cos y)(x\sin y)$$
$\Diamond$

We can also do implicit differentiation.

<u>Ex</u>: Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ where z is implicitly defined by $yz + x\ln y = z^2$.

<u>Sol</u>: Take the derivative of both sides w.r.t. x:
$$y\frac{\partial z}{\partial x} + \ln(y) = 2z\frac{\partial z}{\partial x}$$
Solve for $\frac{\partial z}{\partial x}$: $\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}$

For $\frac{\partial z}{\partial y}$, differentiate both sides w.r.t. y:

$$\left(z + y\frac{\partial z}{\partial y}\right) + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

and solve for $\frac{\partial z}{\partial y}$: $\frac{\partial z}{\partial y} = \frac{z + \left(\frac{x}{y}\right)}{2z - y} = \frac{yz + x}{2yz - y^2}$ ◇

We can, of course, take partial derivatives of functions of 3 or more variables. The procedure is exactly the same: treat all the variables as constants except the one you're taking the derivative with respect to, then differentiate.

Higher-Order Derivatives

Naturally, we can take derivatives of derivatives:

Notation: $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{etc.}$$

notice the change of order here

f_{xy} means "take the x then y derivative"

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ means "take $\frac{\partial}{\partial y}$ of $\frac{\partial f}{\partial x}$"

We can take higher partials as well: third, fourth, etc.

Ex: Find all second partials of $f(x,y) = x^4y^3 - y^4$

Sol: $f_x = 4x^3y^3$ $\qquad f_y = 3x^4y^2 - 4y^3$

$f_{xx} = 12x^2y^3$ $\qquad f_{yy} = 6x^4y - 12y^2$

$$\boxed{f_{xy} = 12x^3y^2 \qquad f_{yx} = 12x^3y^2}$$

SAME! $\quad\Diamond$

The fact that $f_{xy} = f_{yx}$ is no coincidence...

Clairaut's Theorem: Suppose f is defined on a disk D containing the point (a,b). If f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

14.4 (edited) - Differentiability

We will do this in the most general setting.

Def: Let $\vec{F}(x_1,\cdots,x_n) = \langle f_1(x_1,\cdots,x_n),\cdots,f_m(x_1,\cdots,x_n)\rangle$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. The total derivative of $\vec{F}$ is

$$D\vec{F}(x_1,\cdots,x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

<u>Def</u>: Let $\vec{F}(x_1, \cdots, x_n) = \langle f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\rangle$ be defined at $(a_1, \cdots, a_n)$ and all points arbitrarily close to it. Then, we say $\vec{F}$ is <u>differentiable</u> at $\vec{a} = (a_1, \cdots, a_n)$ if $D\vec{F}(\vec{a})$ exists and

$$\lim_{\vec{h} \to \vec{0}} \frac{\|\vec{F}(\vec{a}+\vec{h}) - \vec{F}(\vec{a}) - D\vec{F}(\vec{a})\vec{h}\|}{\|\vec{h}\|} = 0.$$

<u>Theorem</u>: If $\vec{F}$ is differentiable at $\vec{a}$, it is continuous at $\vec{a}$.

A more toned-down version of the differentiable definition is:

<u>Theorem</u>: If f_x & f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b). $(f = f(x,y))$.

<u>14.6g</u>: Gradients

If $f = f(x,y)$, the <u>gradient</u> of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

if $f = f(x,y,z)$,

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

<u>Ex</u>: Find the gradient of $f(x,y,z) = xe^{xyz}$.

<u>Sol</u>:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle e^{xyz} + xyze^{xyz}, x^2ze^{xyz}, x^2ye^{xyz} \right\rangle \quad \diamond$$

14.5 - Chain Rule

<u>(Proper) Chain Rule</u>: Suppose $\vec{F}(x_1, \cdots, x_n) = \langle f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n) \rangle$ and $\vec{G}(y_1, \cdots, y_p) = \langle g_1(y_1, \cdots, y_p), \cdots, g_n(y_1, \cdots, y_p) \rangle$ and that the range of $\vec{G}$ is inside the domain of $\vec{F}$. If $\vec{G}$ is differentiable at $(a_1, \cdots, a_p) = \vec{a}$ and $\vec{F}$ is differentiable at $\vec{G}(\vec{a})$, then

$$\left[D(\vec{F} \circ \vec{G})\right](\vec{a}) = D\vec{F}(\vec{G}(\vec{a})) \cdot D\vec{G}(\vec{a}).$$

Let's reduce this now to the case when the outside function is scalar valued:

<u>Chain Rule</u>: Let $z = f(x_1, \cdots, x_n)$ and

$\vec{G}(y_1, \cdots, y_p) = \langle g_1(y_1, \cdots, y_p), \cdots, g_n(y_1, \cdots, y_p) \rangle = \langle x_1(y_1, \cdots, y_p), \cdots, x_n(y_1, \cdots, y_p) \rangle$.

With conditions as above: $\vec{a} = (a_1, \cdots, a_p)$

$$\frac{\partial z}{\partial y_i}(\vec{a}) = \nabla f(\vec{G}(\vec{a})) \cdot \frac{\partial \vec{G}}{\partial y_i}(\vec{a})$$